Solid/Liquid Washing Theory: Calculations for Nonequilibrium Stages

Calculations for nonequilibrium stages were systematized via material balances for multicomponent mixing, which is applicable in the absence of sorption.
The resulting treatment is quite complex with unequal stages but much simpler when the stages are equal. It allows solving complex solid/liquid washing problems from experimental simple washing data obtained with solute-free wash liquor.

A. TOMIAK

The Ontario Paper Co.
Thorold, Ontario, Canada

SCOPE

The objective of this work was to present a method for stagewise washing calculations for nonequilibrium stages. This is relevant because of the quest for more efficient solid/liquid washing methods and its applicability to solving similar problems.

No sorption and constant volumetric liquor hold-up in the cake at all stages are assumed, which corresponds to so-called displacement washing.

CONCLUSIONS AND SIGNIFICANCE

It was shown that solutions to complex solid/liquid washing problems involving nonequilibrium stages are obtainable from experimental data for the basic case of simple washing, involving the use of solute-free wash liquor under the same operating conditions. They are based on characteristic f^* vs. N curves, where f^* is the normalized average concentration in the cake liquor hold-up and N is the volumetric wash ratio for such tests.

The above provides the mathematical basis for the design and optimization of complex solid/liquid washing systems. Similar approach may be used with advantage in other applications, avoiding the restriction of using particular models to describe the experimental curves.

The method could be used as a subroutine for computerized general material balance programs, thus permitting overall process optimization studies for the whole plant to pinpoint the effect of minimizing the wash liquor usage on subsequent chemical recovery operations, e.g., evaporation.

INTRODUCTION

Recovery of solute present in the liquor entrained in the solids cut from solid/liquid separations is an important operation in chemical engineering. For efficient recovery, involving minimizing the wash liquor usage, countercurrent stages are commonly used. Calculation methods for equilibrium stages are well known and widely used, e.g., in the design of countercurrent decantation systems. Analogous method for nonequilibrium stages are more difficult to handle and less developed.

This work deals with an extension of a calculation method proposed for washing on a belt filter. At first (Tomiak, 1979a) only the special case of countercurrent washing with equal stages was considered; but the treatment was subsequently extended to other flow patterns (Tomiak, 1982a) and further to unequal stages for dynamic filters (Tomiak, 1982b). A phenomenological description of the equal stages case with supporting experimental data was provided by Hermia and Letesson (1982); the general case of unequal stages was treated via the superposition principle of mathematical physics, with an elegant short-cut way of calculating the final liquor concentration in the washed cake (Norden, Viljakainen and Nousiainen, 1982). With increasing concern about more efficient washing and potential widespread use of dynamic filters, the method merits wider recognition. In particular, some basic explanation going beyond purely mathematical aspects is needed.

Satisfactory explanation of the above, while basically straightforward is difficult because it involves use of different symbols to denote various concentrations applied to specific cases of the general treatment defined by two sets of simultaneous equations. The first set describes the washing history of the cake and takes into account the residual hold-up of the original liquor in the cake and the contributions of the various wash liquor additions, while the second set defines the wash iquor concentrations via material balances for the individual stages and depends on the particular flow pattern.

Understanding the basic treatment of the problem is difficult until one grasps the relationship between the characteristic simple washing curve data—corresponding to using solute-free wash liquor at all stages—and the mathematics of the general case. This relationship, stemming from an implicit assumption of additivity of volumes, is not obvious at a glance and appears to be confusing until associated with the general case of multicomponent mixing.

VOLUMETRIC WASH RATIO

The principal variable in washing is the volumetric wash ratio, defined by:

N = volume of wash liquor/volume of cake liquor hold-up

The filter cake solids act as a carrier of the cake liquor holdup, whose volume was taken to be the same at the exit from each stage. This is approximately true in practice and in principle it can be achieved by controlling the filter cake dewatering after each wash liquor application.

The following treatment is limited to material balance considerations and is specifically concerned with translating simple filtration washing data in the form of f^* vs. N curve, characteristic of a given system, to more complex situations, e.g., countercurrent washing, under the same operating conditions (i.e., at a given flow rate, temperature and pressure, and with the same way of wash liquor application).

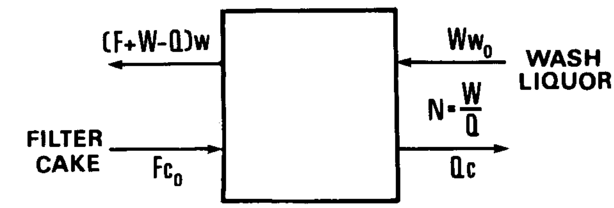

Figure 1. Generalized washing block diagram.

Although various theoretical equations for f^* as a function of N are available, based on different washing models, in general f^* is an experimental curve because the models are not truly predictive and require curve fitting to determine their parameters. Using the theoretical equations is helpful but not necessary.

MULTICOMPONENT MIXING

Multicomponent mixing, assuming additivity of volumes, is defined by the following two equations:

$$\sum \alpha_k a_k = a \qquad (1)$$

$$\sum \alpha_k = 1 \qquad (2)$$

with $\alpha_k = V_k / \Sigma V_k$ being the volumetric fraction of the kth component.

Note that Eq. 2 may be obtained from Eq. by putting $a_k = a$, which corresponds to mixing of the same concentration liquors. Analogous treatment for more complex cases will be used later.

For a binary system, we have:

$$\alpha_1 a_1 + \alpha_2 a_2 = a \qquad (3)$$

$$\alpha_1 + \alpha_2 = 1 \qquad (4)$$

FRACTIONAL RESIDUAL HOLD-UPS AND DISPLACEMENTS

Applying the multicomponent mixing equations to the mixing of the residual original liquor at a concentration c_o and the wash liquor concentrations w_k yields:

$$h^o c_o + \sum d_k w_k = c \qquad (5)$$

$$h^o + \sum d_k = 1 \qquad (6)$$

for the general and

$$h c_o + d w_o = c \qquad (7)$$

$$h + d = 1 \qquad (8)$$

for the binary mixing, where c_o and w_k are the concentrations of the original liquor and the kth wash liquor, respectively, and w_o is the concentration of the original wash liquor.

Based on binary mixing, Appendix 1, any concentration a in a washing system, Figure 1, may be considered to arise from a fractional hold-up of the original liquor in the cake h:

$$h = (a - w_o)/(c_o - w_o) \qquad (9)$$

and a fractional displacement by the wash liquor d:

$$d = (c_o - a)/(c_o - w_o) \qquad (10)$$

with the latter equivalent ot the so-called displacement ratio, widely used in pulp washing calculations in the pulp and paper industry (Perkins, Welsh and Mappus, 1954).

These fractional hold-ups and displacements arise from mixing imposed by the flow of wash liquor through the filter cake and for the same flow conditions, and similar physical properties of the original and wash liquors may be expected to be independent of concentration as was experimentally verified (Hermia and Letesson, 1982). In such cases, knowing any concentration a_o obtainable with solute-free wash liquor ($w_o = 0$) and original liquor concentration c_o allows one to predict that concentration a' would be obtained under the same conditions but with original and wash liquor concentrations c_o' and w_o', respectively, from:

$$a' = \frac{a_o}{c_o}(c_o' - w_o') + w_o' \qquad (11a)$$

$$= c_o' - \frac{c_o - a_o}{c_o}(c_o' - w_o') \qquad (11b)$$

The above equation provides a useful coupling relationship for complex washing systems calculations. One can split such systems into various components, carry out the initial calculations for each of them for the simplest case of solute-free wash liquor, and then combine the results according to the given flow pattern (Tomiak, 1979b).

The concept of fractional displacement may be extended to multistage washing. In the case of unequal stages, Appendix 2, the following equation may be written for the average concentration of the liquor hold-up in the cake discharged from the jth stage:

$$c_j = h_{1j} b_o + d_{1j} w_1^o + d_{2j} w_2^o + d_{3j} w_3^o + \ldots + d_{jj} w_j^o \qquad (12a)$$

which for equal stages may be rewritten as:

$$c_j = h_j^o b_o + d_j w_1^o + d_{j-1} w_2^o + d_{j-2} w_3^o + \ldots + d_1 w_j^o \qquad (12b)$$

where h_{1j} and h_j^o are the residual hold-ups of the original liquor, and d_{ij} denotes the fractional displacement in stage j due to wash liquor addition at stage i, which for equal stages corresponds to the fractional displacement d_i originating from wash liquor addition $i - 1$ stages ahead of the given stage.

The above leads to the following set of equations:

$$c_1 = h_{11} b_o + d_{11} w_1^o \qquad (13a)$$

$$c_2 = h_{12} b_o + d_{12} w_1^o + d_{22} w_2^o \qquad (14a)$$

$$c_3 = h_{13} b_o + d_{13} w_1^o + d_{23} w_2^o + d_{33} w_3^o \qquad (15a)$$

$$\vdots \qquad \vdots$$

and

$$c_1 = h_1^o b_o + d_1 w_1^o \qquad (13b)$$

$$c_2 = h_2^o b_o + d_2 w_1^o + d_1 w_2^o \qquad (14b)$$

$$c_3 = h_3^o b_o + d_3 w_1^o + d_2 w_2^o + d_1 w_3^o \qquad (15b)$$

$$\vdots \qquad \vdots$$

for the two cases.

It is also shown in Appendix 2 that the fractional displacements are defined in terms of the fractional hold-ups by:

$$d_{(j-i)j} = h_{(j-i+1)j} - h_{(j-1)j} \qquad (16)$$

$$d_i = h_{i-1}^o - h_i^o \qquad (17)$$

for the unequal (with $h_{01} = 1$) and equal (with $h_o^o = 1$) stages, respectively.

DETERMINATION OF FRACTIONAL HOLD-UPS

It is shown in Appendix 3 that the fractional hold-ups may be determined from the results of washing tests performed under the given conditions with solute-free wash liquor additions at all stages.

In the case of m unequal stages, m such tests are needed, e.g. for 5 stages one must know:

$h_{11} = f_1^*(N)$

$h_{12} = f_1^*(2N) \quad h_{22} = f_2^*(N)$

$h_{13} = f_1^*(3N) \quad h_{23} = f_2^*(2N) \quad h_{33} = f_3^*(N)$

$h_{14} = f_1^*(4N) \quad h_{24} = f_2^*(3N) \quad h_{34} = f_3^*(2N) \quad h_{44} = f_4^*(N)$

$h_{15} = f_1^*(5N) \quad h_{25} = f_2^*(4N) \quad h_{35} = f_3^*(3N) \quad h_{45} = f_4^*(2N)$

$h_{55} = f_5^*(N)$

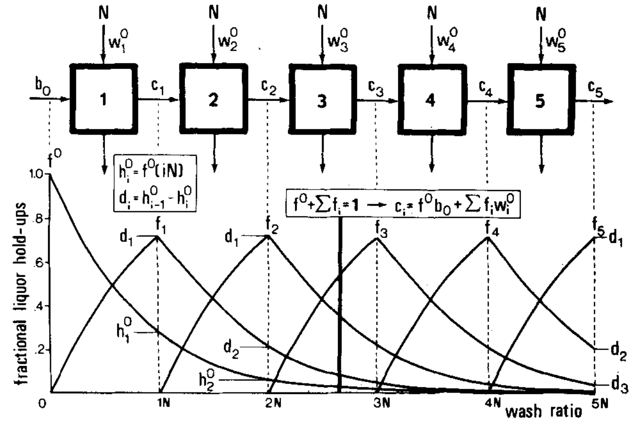

Figure 2. Typical simple washing ($f^o = f^*$) and washing displacement (f_i) curves.

where $f_i^*(jN)$ refers to the simple washing curve obtained with the solute-free wash liquor additions starting with the ith stage and jN is the overall wash ratio due to j wash liquor additions each at a wash ratio N.

For equal stages, a single washing test is needed, for which:

$$h_1^o = f^*(N)$$
$$h_2^o = f^*(2N)$$
$$h_3^o = f^*(3N)$$
$$h_4^o = f^*(4N)$$
$$h_5^o = f^*(5N)$$

The case of equal stages is represented in Figure 2, where $f^o = f^*(N)$ is the characteristic simple washing curve, and f_i's are the washing displacement curves derived from Eq. 17, with all the f_i curves of the same shape but with different origins, each starting at the stage of wash liquor addition.

In the above, the $f^*(jN)$ values are equal to normalized concentrations obtained in the simple washing test with solute-free wash liquor. (They could also be obtained with wash liquor at a concentration w_o by making use of Eq. 9, which amounts to using reduced normalized concentrations defined by the equation.)

SETTING UP SYSTEM'S EQUATIONS

For nonequilibrium stages, the concentrations of liquor holdups in the cake are not uniform and their average values differ from the concentrations of the washings. Note that, due to the mixing in the washings tanks, uniform concentration wash liquors are obtained even though the instantaneous washings concentrations change.

For a system of m stages with the original liquor and wash concentrations b_o and w_o, there are thus $2m$ unknowns in the system's material balance: m average cake liquor hold-up concentrations c_i and m wash liquor concentrations w_i^o. These concentrations are defined by two sets of equations. The first set is given by the fractional hold-up and displacements equations represented by Eqs. 13–15 with the particular values of h and d's defined by the appropriate simple washing curves f^* vs. N. The other set is given by the material balance equations for individual stages, which are obtainable in a straightforward way. (This involves matching the w_i^o wash liquor concentrations with the w_i washings concentrations to conform with a given flowsheet. For example, for countercurrent washing we have $w_i^o = w_{i+1}$.)

For five-stage countercurrent washing, Figure 3, we thus have:

$$c_1 = h_1^o b_o + d_1 w_2 \quad (18)$$

$$c_2 = h_2^o b_o + d_2 w_2 + d_1 w_3 \quad (19)$$

$$c_3 = h_3^o b_o + d_3 w_2 + d_2 w_3 + d_1 w_4 \quad (20)$$

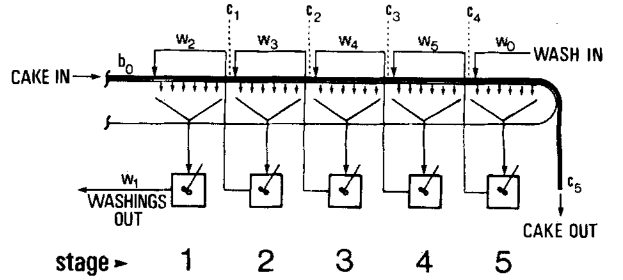

Figure 3. Flowsheet for five-stage countercurrent washing.

$$c_4 = h_4^o b_o + d_4 w_2 + d_3 w_3 + d_2 w_4 + d_1 w_5 \quad (21)$$

$$c_5 = h_5^o b_o + d_5 w_2 + d_4 w_3 + d_3 w_4 + d_2 w_5 + d_1 w_o \quad (22)$$

and

$$b_o - c_1 = N(w_1 - w_2) \quad (23)$$

$$c_1 - c_2 = N(w_2 - w_3) \quad (24)$$

$$c_2 - c_3 = N(w_3 - w_4) \quad (25)$$

$$c_3 - c_4 = N(w_4 - w_5) \quad (26)$$

$$c_4 - c_5 = N(w_5 - w_o) \quad (27)$$

which allows calculating the c_1, c_2, c_3, c_4, c_5 and w_1, w_2, w_3, w_4, w_5 values for the given b_o, w_o, N and known h_i^o and d_i's.

SOLVING THE EQUATIONS

As shown in Appendix 4, the need to solve the equations may be circumvented by two rounds of calculations, the first starting with an assumed trial-and-error value and the second with its corresponding correct value, obtained via binary mixing formula.

In the case of countercurrent washing with solute-free wash liquor, if only the value of the washing loss $C_m = c_m/b_o$ is needed, one can calculate it by the following formulas given by Norden, Viljakainen and Nousiainen:

$$(N - 1 + h_{jj})D_j = \sum_{i=1}^{j-1} D_i[h_{i(j-1)} - h_{ij}] \quad (28)$$

$$S_m = \sum_{j=1}^{m} D_j \quad (\text{with } D_1 = 1) \quad (29)$$

$$C_m = \frac{h_{11}^o - (S_m - 1)(N - 1)}{S_m} \quad (30)$$

where D and S are auxiliary parameters and C_m is the normalized concentration in the washed cake leaving stage m.

Such calculations for five equal stages, Appendix 4, are extended to all stage concentrations.

NOTATION

Small letters denote liquid-phase concentrations in weight/volume units:

a = any concentration
b_o = initial filter cake liquor hold-up concentration
c = filter cake liquor hold-up concentration
w = washings concentration
w_i^o = concentration of wash liquor used at stage i

Capital letters denote flow rates in volume/time units:

F = flow rate of filter cake feed liquor
Q = flow rate of filter cake discharge liquor
W = flow rate of wash liquor and washings

Other symbols:

$C = c/c_o$ = normalized cake liquor hold-up concentration, dimensionless
D = auxiliary parameter, dimensionless

d = fractional displacement by the wash liquor, dimensionless
f^* = $(c - w_o)/(b_o - w_o)$ = reduced normalized concentration in washed cake = simple washing loss, obtained with solute-free wash liquor, dimensionless
h = fractional hold-up of original liquor, dimensionless
i = stage number, dimensionless
j = stage number and number of individual stage wash ratios, dimensionless
m = number of stages, dimensionless
N = W/Q = wash ratio, dimensionless
S_m = auxiliary parameter, dimensionless
V = volume, volume units

Superscripts

^{o} in h^o = fractional hold-up of original liquor remaining in the washed cake (without allowing for the original liquor content in recycled washings)
^{o} in w_i^o = concentration of wash liquor added at stage i (w_i = concentration of the washings leaving stage i)

Subscripts

k = kth component in the mixture of liquids at various solute concentrations
ij = contribution from wash liquor addition at stage i to cake liquor discharged from stage j, at unequal stages
j = residual hold-up of original liquor in cake discharged from stage j, at equal stages
i = displacement of liquor in cake discharged from a stage due to wash liquor addition $i - 1$ stages ahead of it, at equal stages

APPENDIX 1: APPLICATION OF BINARY MIXING EQUATIONS TO WASHING

For the washing system, Figure 1, any concentration within the system results from mixing of the original liquor in the cake at c_o concentration with the wash liquor at w_o concentration. The resultant concentrations fall between c_o and w_o and are given by the binary mixing formulas:

$$hc_o + dw_o = a \qquad \text{(A1-1)}$$

$$h + d = 1 \qquad \text{(A1-2)}$$

where h is the fractional hold-up of the residual original liquor and d is the fractional displacement by the wash liquor.

Thus any concentration a is defined by:

$$a = hc_o + dw_o = h(c_o - w_o) + w_o \qquad \text{(A1-1a)}$$

$$= c_o - d(c_o - w_o) \qquad \text{(A1-1b)}$$

with the proper values of h and d depending on the mixing imposed by the flow of liquor through the filter cake.

For the same mixing, i.e., with the same flows and similar physical properties of the original and wash liquors, h and d are independent of c_o and w_o concentrations, thus allowing predictions of corresponding a' concentrations, which would result from using c_o' original and w_o' wash liquor concentrations. Thus, knowing a particular concentration a resulting from c_o original and w_o wash liquor concentrations, one can calculate the residual liquor hold-up and displacement from:

$$h = (a - w_o)/(c_o - w_o) \qquad \text{(A1-2)}$$

$$d = (c_o - a)/(c_o - w_o) \qquad \text{(A1-3)}$$

and substitute the above into eqs. A1-1a and A1-1b to obtain:

$$a' = \frac{a - w_o}{c_o - w_o}(c_o' - w_o') + w_o' \qquad \text{(A1-4a)}$$

$$= c_o' - \frac{c_o - a}{c_o - w_o}(c_o' - w_o') \qquad \text{(A1-4b)}$$

In the special case of a concentration a_o obtainable with solute-free wash liquor ($w_o = 0$), we have:

$$a' = \frac{a_o}{c_o}(c_o' - w_o') + w_o' \qquad \text{(A1-5a)}$$

$$= c_o' - \frac{c_o - a_o}{c_o}(c_o' - w_o') \qquad \text{(A1-5b)}$$

Detailed explanation of the above, though using different notation, is given in my previous paper (Tomiak, 1974).

APPENDIX 2: APPLICATION OF GENERAL MIXING EQUATIONS TO STAGEWISE WASHING

Before arriving at a general treatment for any number of stages, a specific example of five-stage countercurrent washing shown in Figure 3 will be considered first.

Denoting the fractional residual hold-up of the original liquor in the filter cake discharged from stage j by h_{1j} and the fractional displacement in it due to wash liquor addition at stage i by d_{ij}, we have:

$$c_1 = h_{11}b_o + d_{11}w_2 \qquad \text{(A2-1a)}$$

$$c_2 = h_{12}b_o + d_{12}w_2 + d_{22}w_3 \qquad \text{(A2-2a)}$$

$$c_3 = h_{13}b_o + d_{13}w_2 + d_{23}w_3 + d_{33}w_4 \qquad \text{(A2-3a)}$$

$$c_4 = h_{14}b_o + d_{14}w_2 + d_{24}w_3 + d_{34}w_4 + d_{44}w_5 \qquad \text{(A2-4a)}$$

$$c_5 = h_{15}b_o + d_{15}w_2 + d_{25}w_3 + d_{35}w_4 + d_{45}w_5 + d_{55}w_o \qquad \text{(A2-5a)}$$

It may be noted that $d_{jj}, d_{(j-1)j}, d_{(j-2)j}, \ldots$ denote the fractional displacements in the liquor hold-up in the filter cake discharged from stage j due to wash liquor additions at the stage, one stage ahead of it, two stages ahead of it, etc. In the case of equal stages, these fractional displacements are independent of the stage location but depend on the relative position of the wash liquor addition point. Thus, all fractional displacements due to wash liquor additions at any stage will be the same, all fractional displacements due to wash liquor additions one stage ahead of any stage will be the same, all fractional displacements due to wash liquor additions two stages ahead of any stage will be the same, etc. Therefore, $d_{11} = d_{22} = d_{33} = d_{44} = d_{55} = d_1$, $d_{12} = d_{23} = d_{34} = d_{45} = d_2$, $d_{13} = d_{24} = d_{35} = d_3$, $d_{14} = d_{25} = d_4$ and $d_{15} = d_5$ for the five-stage countercurrent washing, where d_i denotes the fractional displacements originating from wash liquor addition $i - 1$ stages ahead of a stage. Using such simplified notation, Eqs. A2-1a to A2-5a may be rewritten for equal stages as:

$$c_1 = h_1^o b_o + d_1 w_2 \qquad \text{(A2-1b)}$$

$$c_2 = h_2^o b_o + d_2 w_2 + d_1 w_3 \qquad \text{(A2-2b)}$$

$$c_3 = h_3^o b_o + d_3 w_2 + d_2 w_3 + d_1 w_4 \qquad \text{(A2-3b)}$$

$$c_4 = h_4^o b_o + d_4 w_2 + d_3 w_3 + d_2 w_4 + d_1 w_5 \qquad \text{(A2-4b)}$$

$$c_5 = h_5^o b_o + d_5 w_2 + d_4 w_3 + d_3 w_4 + d_2 w_5 + d_1 w_o \qquad \text{(A2-5b)}$$

Equations A2-1a to A2-5a and A2-1b to A2-5b refer to the special case of countercurrent washing for which the wash liquors added at a stage originate from the stage immediately following it, with the usual way of using subscripts to denote the stage of origin. Thus, w_2, w_3, w_4, w_5, w_o denote the wash liquors added at stage 1,2,3,4,5. Since in principle different routes for the washings may be used, it is best to base the subscripts used for the w's on their destinations. Thus, denoting the wash liquor added to stage i by w_i^o, one can rewrite the equations in the general form shown in Eqs. 13a to 15a and 13b to 15b in the main text.

Due to their complementary character, the fractional displacements may be defined in terms of the fractional hold-ups. To do so, one requires a set of equations corresponding to Eq. 6 of the main text. This may be obtained by substituting $w_i = b_o$ and noting that $c_i = b_o$ in such case, since it corresponds to mixing of the same liquor. Thus, in the case of Eq. A2-1a we have:

$$h_{11} + d_{11} = 1 \qquad \text{(A2-6a)}$$

from which

$$d_{11} = 1 - h_{11} \qquad \text{(A2-6b)}$$

which for any stage j may be generalized as

$$d_{jj} = 1 - h_{jj} \qquad \text{(A2-6c)}$$

Similarly, in the case of Eq. A2-2a we have:

$$h_{12} + d_{12} + d_{22} = 1 \qquad \text{(A2-7a)}$$

$$d_{12} = 1 - h_{12} - d_{22} = h_{22} - h_{12} \qquad \text{(A2-7b)}$$

$$(\text{with } d_{22} = 1 - h_{22})$$

$$d_{(j-1)j} = h_{jj} - h_{(j-1)j} \qquad \text{(A2-7c)}$$

and for Eq. 2A-3a:

$$h_{13} + d_{13} + d_{23} + d_{33} = 1 \qquad \text{(A2-8a)}$$

$$d_{13} = 1 - h_{13} - d_{23} - d_{33} = h_{23} - h_{13} \qquad \text{(A2-8b)}$$

$$(\text{with } d_{23} = h_{33} - h_{23} \quad \text{and } d_{33} = 1 - h_{33})$$

$$d_{(j-2)j} = h_{(j-1)j} - h_{(j-2)j} \qquad \text{(A2-8c)}$$

with the bracketed expressions given by the generalized Eqs. 2A-6c and 2A-7c.

Repeating the treatment, one finds that in general we have:

$$d_{(j-i)j} = h_{(j-i+1)j} - h_{(j-i)j} \qquad \text{(A2-9)}$$

By a similar reasoning applied to the case of equal stages, represented by Eqs. A2-1b to A2-5b we have:

$$h_1^o + d_1 = 1 \qquad \text{(A2-10a)}$$

$$d_1 = 1 - h_1^o \qquad \text{(A2-10b)}$$

for the first stage,

$$h_2^o + d_2 + d_1 = 1 \qquad \text{(A2-11a)}$$

$$d_2 = 1 - h_2^o - d_1 = h_1^o - h_2^o \qquad \text{(A2-11b)}$$

$$(\text{with } d_1 = 1 - h_1^o)$$

for the second stage,

$$h_3^o + d_3 + d_2 + d_1 = 1 \qquad \text{(A2--12a)}$$

$$d_3 = 1 - h_3^o - d_2 - d_1 = h_2^o - h_3^o \qquad \text{(A2-12b)}$$

$$(\text{with } d_2 = h_1^o - h_2^o \text{ and } d_1 = 1 - h_1^o)$$

for the third stage and in general:

$$d_i = h_{i-1}^o - h_i^o \qquad \text{(A2-13)}$$

for the ith stage.

APPENDIX 3: DETERMINATION OF FRACTIONAL HOLD-UPS

The fractional hold-ups may be obtained from the results of simple washing tests performed with solute-free wash liquor, as illustrated below for five countercurrent stages.

For the general case of unequal stages, represented by Eqs. 13a to 15a of the main section and constant wash ratio at each stage, we have for the solute-free wash liquor additions starting from the first stage, i.e., with $w_i^o = 0$:

$$h_{11} = c_1^o/b_o = f_{11}^* = f_1^*(N) \qquad \text{(A3-1a)}$$

$$h_{12} = c_2^o/b_o = f_{12}^* = f_1^*(2N) \qquad \text{(A3-2a)}$$

$$h_{13} = c_3^o/b_o = f_{13}^* = f_1^*(3N) \qquad \text{(A3-3a)}$$

$$h_{14} = c_4^o/b_o = f_{14}^* = f_1^*(4N) \qquad \text{(A3-4a)}$$

$$h_{15} = c_5^o/b_o = f_{15}^* = f_1^*(5N) \qquad \text{(A3-5a)}$$

where c_i^o are the concentrations obtained in stage i as a result of such test under the given conditions. The fractional hold-ups correspond thus to normalized concentrations c_i^o/b_o at iN wash ratios, defined by the experimental f_1^* simple washing curve for the first stage.

By analogous treatment one obtains for the other stages:

$$h_{22} = f_2^*(N)$$

$$h_{23} = f_2^*(2N) \quad h_{33} = f_3^*(N)$$

$$h_{24} = f_2(3N) \quad h_{34} = f_3(2N) \quad h_{44} = f_4^*(N)$$

$$h_{25} = f_2^*(4N) \quad h_{35} = f_3^*(3N) \quad h_{45} = f_4^*(2N) \quad h_{55} = f_5^*(N)$$

where f_i^* are the experimental simple washing curves for stage i, obtained with solute-free wash liquor additions starting from stage i. Note that with m stages, the characteristic simple washing curves must be obtained up to $(m - i + 1)N$ wash ratios to reflect the contributions of the wash liquor additions at a given stage to the stage and all stages following it.

For equal stages we have $f_1^* = f_2^* = f_3^* = f_4^* = f_5^*$, i.e., a single simple washing curve f^* defines all the hold-ups, with:

$$h_i^o = f^*(iN) \qquad \text{(A3-6)}$$

APPENDIX 4: SOLVING EQUATIONS FOR NONEQUILIBRIUM STAGES

The need to solve the two sets of equations defining nonequilibrium stagewise washing may be circumvented by making use of direct calculations for an assumed value of one of the variables and then correcting the numerical values thus obtained to the proper w_o concentration by the following transformation, based on Eq. 9 of the main section:

$$(a - w_o)/(b_o - w_o) = (a' - w_o')/(b_o - w_o') \qquad \text{(A4-1)}$$

where a' and w_o' are the concentrations calculated and a is the corresponding concentration obtainable with wash liquor concentration w_o.

For example, in case of the five-stage countercurrent washing, the following equations may be obtained by manipulating Eqs. 18 to 27 of the main section:

$$w_2 = (Nw_1 - d_1b_o)/(N - d_1) \qquad \text{(A4-2)}$$

$$w_3 = [(N - d_1 + d_2)w_2 - d_2b_o]/(N - d_1) \qquad \text{(A4-3)}$$

$$w_4 = [(N - d_1 + d_2)w_3 - (d_2 - d_3)w_2 - d_3b_o]/(N - d_1) \qquad \text{(A4-4)}$$

$$w_5 = [(N - d_1 + d_2)w_4 - (d_2 - d_3)w_3 - (d_3 - d_4)w_2 - d_4b_o]/(N - d_1) \qquad \text{(A4-5)}$$

$$w_o = [(N - d_1 + d_2)w_5 - (d_2 - d_3)w_4 - (d_3 - d_4)w_3 - (d_4 - d_5)w_2 - d_5b_o]/(N - d_1) \qquad \text{(A4-6)}$$

and a first round of calculations carried out for an assumed high trial-and-error value of $w_1 = w_1'$ to obtain its corresponding $w_o = w_o'$ value. Repeating the calculations for the proper w_1 value obtained via the binary hold-up formula based on Eq. A4-1 with $a = w_1$:

$$w_1 = \frac{w_1' - w_o'}{b_o - w_o'}(b_o - w_o) + w_o \qquad \text{(A4-7)}$$

for the proper w_o value yields the correct w_2, w_3, w_4 and w_5 values, knowing which corresponding c_1, c_2, c_3, c_4 and c_5 may be calculated from Eqs. 23 to 27 of the main section.

If only the value of the washing loss $C_5 = c_5/b_o$ is needed, one can calculate it for the case of solute-free wash liquor from the formulas derived for unequal stages by Norden, Viljakainen and Nousiainen, using the following procedure:

Calculate D_j values from:

$$(N - 1 + h_{jj})D_j = \sum_{i=1}^{j-1} D_i[h_{i(j-1)} - h_{ij}] \qquad \text{(A4-8)}$$

Calculate S_m from:

$$S_m = \sum_{j=1}^{m} D_j \quad (\text{with } D_1 = 1) \qquad \text{(A4-9)}$$

Calculate the loss from:

$$C_m = \frac{h_{11} - (S_m - 1)(N - 1)}{S_m} \qquad \text{(A4-10)}$$

Applying the above as an example to five countercurrent stages, we have:

$$j = 2 \quad (N - 1 + h_{22})D_2 = h_{11} - h_{12} \qquad \text{(A4-11)}$$

$$j = 3 \quad (N - 1 + h_{33})D_3 = h_{12} - h_{13} + D_2(h_{22} - h_{23}) \qquad \text{(A4-12)}$$

$$j = 4 \quad (N - 1 + h_{44})D_4 = h_{13} - h_{14} + D_2(h_{23} - h_{24}) + D_3(h_{33} - h_{34}) \qquad \text{(A4-13)}$$

$$j = 5 \quad (N - 1 + h_{55})D_5 = h_{14} - h_{15} + D_2(h_{24} - h_{25}) + D_3(h_{34} - h_{35}) + D_4(h_{44} - h_{45}) \qquad \text{(A4-14)}$$

which for equal stages may be rewritten as:

$$(N - 1 + h_1^o)D_2 = h_1^o - h_2^o \qquad \text{(A4-11a)}$$

$$(N - 1 + h_1^o)D_3 = h_2^o - h_3^o + D_2(h_1^o - h_2^o) \qquad \text{(A4-12a)}$$

$$(N - 1 + h_1^o)D_4 = h_3^o - h_4^o + D_2(h_2^o - h_3^o) + D_3(h_1^o - h_2^o) \qquad \text{(A4-13a)}$$

$$(N - 1 + h_1^o)D_5 = h_4^o - h_5^o + D_2(h_3^o - h_4^o) + D_3(h_2^o - h_3^o) + D_4(h_1^o - h_2^o) \qquad \text{(A4-14a)}$$

from which

$$S_5 = 1 + D_2 + D_3 + D_4 + D_5 \qquad \text{(A4-15)}$$

may be obtained and finally

$$C_5 = \frac{h_1^o - (D_2 + D_3 + D_4 + D_5)(N - 1)}{1 + D_2 + D_3 + D_4 + D_5} \qquad \text{(A4-16)}$$

calculated.

If the complete material balance data are needed in the case of equal stages, one may proceed as follows:

Calculate the normalized washings concentrations $W_i = w_i/b_o$ using the following formulas, similar to those for calculating the D_j's:

$$W_1 = (1 - C_m)/N \qquad \text{(A4-17)}$$

$$W_2 = (h_1^o - C_m)/(N - 1 + h_1^o) \qquad \text{(A4-18)}$$

$$W_3 = (h_2^o - C_m + d_2W_2)/(N - 1 + h_1^o) \qquad \text{(A4-19)}$$

$$W_4 = (h_3^o - C_m + d_3W_2 + d_2W_3)/(N - 1 + h_1^o) \qquad \text{(A4-20)}$$

$$\vdots$$

Calculate the remaining normalized average concentrations of the cake liquor hold-ups $C_i = c_i/b_o$ from:

$$C_1 = C_m + NW_2 \qquad \text{(A4-21)}$$

$$C_2 = C_m + NW_3 \qquad \text{(A4-22)}$$

$$C_3 = C_m + NW_4 \qquad \text{(A4-23)}$$

$$C_4 = C_m + NW_5 \qquad \text{(A4-24)}$$

$$\vdots$$

TABLE 1. NUMERICAL EXAMPLE

Given		Calculated (Auxiliary)
1.1	N	
0.2327	$h_1^o = (1 + N)e^{-2N}$	*Based on* $f^* = (1 + N)e^{-2N}$,
0.0393	$h_2^o = (1 + 2N)e^{-4N}$	assumed to represent the
0.0059	$h_3^o = (1 + 3N)e^{-6N}$	experimental simple filt-
0.0008	$h_4^o = (1 + 4N)e^{-8N}$	ration washing data for
0.0001	$h_5^o = (1 + 5N)e^{-10N}$	the system.
	$d_2 = h_1^o - h_2^o$	(0.1934)
	$d_3 = h_2^o - h_3^o$	(0.0334)
	$d_4 = h_3^o - h_4^o$	(0.0050)
	$d_5 = h_4^o - h_5^o$	(0.0007)
	$a = N - 1 + h_1^o$	(0.3327)
	$D_2 = d_2/a$	(0.5813)
	$D_3 = (d_3 + d_2D_2)/a$	(0.4384)
	$D_4 = (d_4 + d_3D_2 + d_2D_3)/a$	(0.3284)
	$D_5 = (d_5 + d_4D_2 + d_3D_3 + d_2D_4)/a$	(0.2459)
	$S_5 = 1 + D_2 + D_3 + D_4 + D_5$	(2.5941)
	$C_5 = [a - S_5(N - 1)]/S_5$	0.0282
	$W_1 = (1 - C_5)/N$	0.8834
	$W_2 = (h_1^o - C_5)/a$	0.6145
	$W_3 = (h_2^o - C_5 + d_2W_2)/a$	0.3904
	$W_4 = (h_3^o - C_5 + d_3W_2 + d_2W_3)/a$	0.2214
	$W_5 = (h_4^o - C_5 + d_4W_2 +$	0.0948
	$d_3W_3 + d_2W_4)/a$	0.0948
	$C_1 = C_5 + NW_2$	0.7042
	$C_2 = C_5 + NW_3$	0.4577
	$C_3 = C_5 + NW_4$	0.2718
	$C_4 = C_5 + NW_5$	0.1325

An example of such calculations for the five-stage countercurrent washing system shown in Figure 3 and $f^* = (1 + N)e^{-2N}$ is given in Table 1 for $N = 1.1$. [The equation for f^* corresponds to the perfect mixing cells in series model with two mixing cells (Tomiak, 1973).]

NOTE: The word "stage was used here for what is called "washing zone" in pulp and paper terminology. The distinction between the two terms is important for multistage washing systems using multizone washing at individual stages.

LITERATURE CITED

Hermia, J., and Ph. Letesson, "The Universal Washing Curve and Its Application to Multi-staged Countercurrent Washing Processes," Preprints of World Filtration Congress III, Philadelphia, 426 (1982).

Norden, H. V., E. Viljakainen, and H. Nousiainen, "Calculation of the Efficiency of Multizone Pulp Washers Using a Mass Transfer Model and the Superposition Principle," *J. of Pulp and Paper Sci.*, 8(**2**), TR 21–26 (June, 1982).

Perkins, J. K., H. S. Welsh, and J. H. Mappus, "Brown Stock Washing Efficiency: Displacement Ratio Method of Determination," *TAPPI*, **37**(3), 83 (1954).

Tomiak, A., "Predict Performance of Belt Filter Washing," *Chem. Eng.*, 143 (April 23, 1979a).

Tomiak, A., "Solve Complex Washing Problems," *Chem. Eng.*, 109 (Oct. 8, 1979b).

Tomiak, A., "Practical Application of Filtration Washing Theory," Preprints of World Filtration Congress III, Philadelphia, 436 (1982a).

Tomiak, A., "Washing Loss Calculations for Dynamic Filters," *Ind. Eng. Process Des. Dev.*, **21**, 500 (1982b).

Tomiak, A., "Purging Calculations for Cascades of Mixing Stages," *Can. J. Chem. Eng.*, **52**, 502 (1974).

Tomiak, A., "Theoretical Recoveries in Filter Cake Reslurrying and Washing, *AIChE J.*, **19**, 76 (1973).

Manuscript received October 12, 1982; revision received February 8, and accepted March 9, 1983.